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Hyperspin Manifolds and the Space Problem of Weyl

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A recent criticism of the claimed existence of torsion-free connections compatible with a hypermanifold structure in Finkelstein's sense is reinforced by relating the problem to the space problem of Weyl as generalized by Cartan and Freudenthal. Some historical remarks concerning the development of the latter are also included.

In this comment, I reinforce a recent criticism expressed by Borowiec (1988) concerning a claim made by Holm (1986) that there exists a torsion-free connection preserving the "hypermanifold structure" as introduced by Finkelstein (1986). The group-theoretic aspect of the problem has been indicated, but not been sufficiently exploited, in Borowiec (1988); so I want merely to point out here that the problem is answered already—in the negative, as correctly conjectured by Borowiec (1988)—by the solution, due to Cartan (1923), Freudenthal (1960), and Kobayashi and Nagano (1965), of the famous "space problem of H. Weyl" [*Weylsches Raumproblem*; see Weyl (1923)] in its generalized versions [Cartan (1922), Freudenthal (1960), Kobayashi and Nagano (1965); also see Klingenberg (1959), and Kobayashi and Nomizu (1963, 1969)].

Just as a (pseudo) Riemannian structure on an N-dimensional differentiable manifold M is given by a nondegenerate symmetric tensor field of type $(0, 2)$, a hypermanifold structure in Finkelstein's sense is given by a symmetric tensor field g of type $(0, n)$, $n > 2$, nondegenerate in a suitable sense.

A reformulation in terms of G-structures on manifolds was given already in Borowiec (1988). One chooses a standard form η for a symmetric type $(0, n)$ tensor in R^N (I do not enter into the classification problem of standard forms) and calls a G_η-frame at $x \in M$ any linear frame at x with

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respect to which the components of the given tensor field at x take the standard form η. These G_η-frames are related by linear substitutions from the group $GL(N, R)$ which leave the standard form η invariant, forming a matrix subgroup $G_\eta \subset GL(N, R)$.

The question now is whether one can find a covariant derivative ∇ on M satisfying $\nabla g = 0$ and being torsion-free whenever a sufficiently generic g is chosen. A reformulation is whether there is a torsion-free connection defined in any principal G_η bundle $P(G_\eta, M)$ formed by G_η-frames at all $x \in M$: its connection form has to take values in the matrix Lie algebra of the matrix group G_η. The analogous problem has been studied, and solved, for all Lie subgroups $G \subset GL(N, R)$, in the works cited above. The answer is that torsion-free connections exist for any principal G-subbundle of the bundle $F(M)$ of all linear frames if and only if G is one of the following subgroups of $GL(N, R)$ (we assume $N \geq 3$):

(i) $GL(N, R)$
(ii) Subgroups of $GL(N, R)$ leaving a one-dimensional subspace invariant and satisfying a certain trace condition.
(iii) $SL(N, R)$
(iv) $O(p, q)$, where $p+q=N$
(v) $CO(p, q)=$ (iv) extended by scale transformations
(vi) $CSp(4, R)=$ symplectic group in four dimensions extended by scale, if $N=4$.

The groups G_η considered here are not included in this list. This reinforces the analysis of Borowiec (1988).

As a historical remark, I add that Weyl's original problem was restricted in two ways: he required (a) $G \subset SL(N, R)$, and (b) the connection is determined *uniquely.*

Weyl proved that only $O(p, q)$ is allowed in this case, and his proof was different from all the following ones. Weyl mentioned that restriction (a) is not necessary for $N \geq 3$, but did not prove this explicitly. Cartan kept restriction (a), but did not require (b). His list of possible results included too much [namely, $Sp(N, R)$ for $N \geq 4$] because he exploited only necessary conditions. Freudenthal dropped both (a) and (b) and seems to be the first to obtain complete results for all $N \geq 2$. Klingenberg formulated the result in the modern way, keeping restriction (b) only, but did not exclude $N=2$, where more than the orthogonal groups are allowed. Kobayashi and Nomizu (1963) gave Klingenberg's formulation, including the restriction $N \geq 3$. In Kobayashi and Nomizu (1969) these authors already quoted the result of Kobayashi and Nagano (1965), which agrees with Freudenthal, except that only $N \geq 3$ is considered and for $N=4$ the group $CSp(4, R)$ is omitted, due to a numerical error. From Cartan on, all proofs use the classification of Lie algebras and their representations; Freudenthal expressed the hope that his ideas to simplify Cartan's proof (which is of considerable length) can be finally developed into a direct proof not going through the classification, and regrets not having succeeded completely. I know of no further progress in this direction.

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